FORCING AT SUCCESSORS OF SINGULAR CARDINALS AND THE SEARCH OF NEW FORCING AXIOMS (VIDEO COURSE MATERIAL)

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These are notes accompanying Mirna Dzamonja's video course which can be found at `https://www.logiqueconsult.eu/forcing-at-singular-cardinals`. This file contains the entire material sectioned according to the video lectures (seven lectures in total). This is with the exception of Section "Preserving Cardinals" You can find separate files corresponding to the lectures next to the video's on the course page.

1. Singular Cardinals (Video lecture 1)

"Singular cardinals appeared on the mathematical landscape two years before they were defined."

Who said this?

The word "singular" in mathematics typically refers to "exceptional", "pathological" or a special case that is difficult to handle. For example the *singular* point of the function $f(x) = \frac{1}{x-1}$ is at $x = 1$. In set theory, *singular cardinals* (see the definition below) are certainly difficult to handle (this is the topic of this course), but they are certainly not rare or exceptional. If they are considered pathological, then only because successor cardinals receive much more attention in main stream set theory research. For example, it is consistent with ZFC that all limit cardinals are singular. On the other hand, if there is a limit cardinal κ that is not singular, then the set of singular cardinals below it is stationary, and unless κ is Mahlo, it is closed and unbounded. In this sense singular cardinals are not less ubiquotous than the so-called *regular cardinals*.

The cofinality of a cardinal κ, $\mathrm{cf}(\kappa)$ is the smallest cardinal $\lambda \leqslant \kappa$ such that there exists an unbounded sequence $(s_\alpha)_{\alpha<\lambda} \subset \kappa$, i.e. for all $\beta < \kappa$ there is $\alpha < \lambda$ with $s_\alpha \geqslant \beta$:

1.1. **Definition.** $\mathrm{cf}(\kappa) = \min\{\theta \mid \text{for some increasing } h\colon \theta \to \kappa, \sup h''\theta = \kappa\}$

A cardinal κ is said to be *singular*, if $\mathrm{cf}(\kappa) < \kappa$ and *regular*, if $\mathrm{cf}(\kappa) = \kappa$. All cardinals are either singular or regular, because obviously it always holds that $\mathrm{cf}(\kappa) \leqslant \kappa$.

Successor cardinals $\kappa = \lambda^+$	Limit cardinals $\forall \lambda < \kappa (\kappa > \lambda^+)$
Regular cardials $\mathrm{cf}(\kappa) = \kappa$	Singular cardinals $\mathrm{cf}(\kappa) < \kappa$

Under the Axiom of Choice all successor cardinals are regular:

1.2. **Theorem** (Hausdorff 1908 (?))**.** *If the Axiom of Choice holds, then every successor cardinal is regular.*

On the other hand, it is easy to construct a singular limit cardinal. For example take the limit $\aleph_\omega = \sup_{n<\omega} \aleph_n$. The existence of regular limit cardinals (large cardinals) does not follow from ZFC, but is a common assumption in set theory.

1.3. **Definition.** If a cardinal is both a limit cardinal and regular, then it is called weakly inaccessible.

Without the axiom of choice we can define that an ordinal number α is a cardinal, if it is not in a bijection with any smaller ordinal. But without the Axiom of Choice (AC) we do not have the theorem that every set has a cardinality. In fact a "small amount" of the AC is consistent with the opposite:

1.4. **Theorem** (Gitik 1980)**.** *Given some large cardinal assumptions it is consistent that all infinite cardinals have countable cardinality and the Axiom of Dependent Choice holds.*

In the model provided by Gitik for this theorem the real numbers don't have a cardinality.

12min 30sec mark of Lecture 1

König's Lemma:

1.5. **Theorem** (König's Lemma)**.** *For any $\kappa \geqslant \aleph_0$, we have $\kappa^{\mathrm{cf}(\kappa)} > \kappa$.*

Another formulation of the same Lemma:

1.6. **Theorem** (König's Lemma General)**.** *Given two sequences $\langle \kappa_i \mid i \in I \rangle$, $\langle \lambda_i \mid i \in I \rangle$ of cardinals such that $\forall i (\kappa_i < \lambda_i)$, then $\sup \kappa_i < \prod_{i \in I} \lambda_i$.*

Theorem 1.5 follows from 1.6. Shelah developed PCF theory which is a theory about products and inequalities of the form as in Theorem 1.6. A proof of Theorem 1.6 can be found in the book by Hajnal and Hamburger [1].

1.7. **Exercise.** Either come up with the proof of 1.5 from 1.6 or study it from [1].

From now on we assume the Axiom of Choice. But even though the AC gives every set a cardinality, it remains undecided which precise cardinal value is given to certain sets. In particular for all κ there can consistently be many different values for 2^κ.

1.8. **Lemma.** *If κ is a limit cardinal, then $2^\kappa = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$ where $2^{<\kappa} = \sup\{2^\lambda \mid \lambda < \kappa\}$.*

Proof. Exercise (see also [2, Thm 5.16]). □

The Continuum Hypothesis (CH) is the statement that $2^{\aleph_0} = \aleph_1$. The Generalised Continuum Hypothesis (GCH) is the assumption that for all infinite cardinals κ we have $2^\kappa = \kappa^+$. Note that GCH implies CH, and not-CH implies not-GCH. We say that "GCH holds below κ" if for all $\lambda < \kappa$, $2^\lambda = \lambda^+$. More generally, *the continuum function* is the function Exp from cardinals to cardinals such that $Exp(\kappa) = 2^\kappa$. Thus e.g. the GCH is the statement that for all κ, $Exp(\kappa) = \kappa^+$.

1.9. **Corollary.**
(1) If κ is a limit cardinal and the GCH holds below κ, then $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.
(2) If κ is a singular cardinal and the continuum function is eventually constant below κ, then $2^\kappa = 2^{<\kappa}$.

Note that in the assumption of 2. above, when the continuum function is constant θ, then this constant must be $\theta > \kappa$. If it was $\theta < \kappa$, then we would have $2^\theta \leqslant \theta$ which is a contradiction. If $\theta = \kappa$, then there is a regular λ with $\mathrm{cf}(\kappa) \leqslant \lambda < \kappa$ and $2^\lambda = \kappa$ which is a contradiction, because now by König's Lemma $2^\lambda = (2^\lambda)^{\mathrm{cf}(k)} = \kappa^{\mathrm{cf}(\kappa)} > \kappa$.

This shows that the value 2^κ for singular κ can depend on the values α^λ for $\lambda < \kappa, \alpha \leqslant \kappa$. What about regular cardinals? Let's look at the broad picture. Kurt Gödel proved in 1937 that both CH and GCH are consistent with ZFC, but predicted that they are also both consistently false. In 1963 Paul Cohen proved that.

1.10. **Theorem** (Gödel 1937)**.** *The consistency of ZFC implies the consistency of GCH.*

1.11. **Theorem** (Cohen 1963)**.** *The consistency of ZFC implies the consistency of the negation of CH.*

Proof. Given the forcing notion $\mathbb{P} = \{f\colon \omega \times \omega_2 \to 2 \mid dom(f) \text{ is finite}\}$, where $p \leqslant q \iff p \subset q$ (Cohen's original notation), we have $V[G] \models 2^{\aleph_0} = \aleph_2$. $\square$

30min 00sec mark Lecture 1

Later a much stronger theorem was proved by Easton:

1.12. **Theorem** (Easton 1968)**.** *Suppose that $V \models GCH$ (for simplicity) and A is a class of regular cardinals. Further suppose $F\colon A \to Card$ is a function satisfying*

(1) if $\kappa \leqslant \lambda$ are in A, then $F(\kappa) \leqslant F(\lambda)$ and
(2) $\forall \kappa \in A\ \mathrm{cf}(F(\kappa)) > \kappa$

Then there is a cardinal and cofinality preserving forcing extension $V[G]$ such that $\forall \kappa \in A\big((2^\kappa)^{V[G]} = F(\kappa)\big)$.

By a class we always mean $\{x \mid \varphi(x, z)\}$ for some first-order formula φ with parameters z. A class function is considered as a class of pairs. In the above theorem one can choose without loss of generality for A to be the class of all regular cardinals.

Note that by the discussion above, singular cardinals cannot be included into A without losing generality.

Preserving Cardinals

The video for this section is lost.
I will write the material for it anyway.
At this moment it's a draft.

Cohen's case: $\mathrm{Con}(2^{\aleph_0} = \aleph_2)$

We don't want the cardinals to change. We want that every ordinal of which V thinks that it is a cardinal, that $V[G]$ also thinks of it that it is a cardinal.

Cardinal are preserved due to the chain condition. CCC - all antichains are countable, Two conditions are incompatible ($p \perp q$) if there is no r such that $r \geqslant p$ and $r \geqslant q$. Using this ccc we can prove that if there is a bijection from $(\aleph_0)^V$ to $(\aleph_2)^V$ in $V[G]$ then such bijection would exist already in V.

Kunen's book: chapter VII

1.13. **Definition.** (1) A forcing notion P is κ-cc, if every antichain is of size $< \kappa$. *ccc*$=\aleph_1 - cc$.
(2) $\mathbb{P}$ is $(< \kappa)$-closed if every increasing sequence of size $< \kappa$ has an upper bound.

1.14. **Fact** (Cohen)**.** *κ-cc forcing preserves cardinals $\geqslant \kappa$. $(< \kappa)$-closed forcing preserves cardinals $\leqslant \kappa$.*

2. Proof of Easton Theorem (Video lecture 2)

Recall that we start from a model which satisfies GCH (for example $V = L$), A a class of regular cardinals and a class-function $F\colon \mathcal{A} \to Card$. We want in $V[G]$ to have $(\forall \kappa \in A)(2^\kappa = F(\kappa))$. If $A = \{\kappa\}$, then this can be done with a version of Cohen's forcing:

$$\mathbb{P}_\kappa = \{f\colon \kappa \times F(\kappa) \to 2 : |\operatorname{dom}(f)| < \kappa\}$$

ordered by $\subset$ is the desired forcing notion.

We want to go beyond that to prove this for larger sets A and even class-size A. For this purpose we introduce *Easton product.*

Suppose $A = \{\aleph_0, \aleph_1\}$ and $F(\aleph_0) = F(\aleph_1) = \aleph_2$. If we first were to force $2^{\aleph_0} = \aleph_2$, then in the extension we would no longer have GCH below $\aleph_1$ and the forcing $\mathbb{P}_{\aleph_1}$ would no longer have the $\aleph_2$-chain-condition. So instead, it is better to go the otherway around: first force $2^{\aleph_1} = \aleph_2$ and only then $2^{\aleph_0} = \aleph_2$. But if A is large, then this is not so straightforward and this is the motivation behind the Easton product.

Let $\mathbb{P}^* = \prod_{\kappa \in A} \mathbb{P}_\kappa$. For $p \in \mathbb{P}^*$, $\operatorname{sprt}(p) = \{\kappa \mid p(\kappa) \neq \varnothing\}$. Let

$$\mathbb{P} = \{p \in \mathbb{P}^* \mid \text{for every regular } \theta, |\operatorname{sprt}(p) \cap \theta| < \theta\}.$$

Every element of p is a function with values 0 and 1, and entries of the type $(\kappa, \alpha, \beta) \mapsto p(\kappa)(\alpha, \beta)$.

Any $\mathbb{P}$-generic G gives rise to a $\mathbb{P}_\kappa$-generic G_κ which produces $F(\kappa)$ new subsets of κ for each $\kappa \in A$.

2.1. **Exercise.** Prove that the number of new subsets of κ of size κ added by $\mathbb{P}$ is exactly $F(\kappa)$.

14min 00sec mark Lecture 2

It is hard to count the anti-chains of this forcing.

2.2. **Lemma.** *P preserves cardinalities and cofinalities.*

The proof starts here and continues in the next lecture.

Let λ be any regular cardinal. Split $\mathbb{P}$ into two parts: for each condition $p \in \mathbb{P}$, let

$$p^{\leqslant\lambda} = p \restriction \{(\kappa,\alpha,\beta) \mid \kappa \leqslant \lambda\}$$

and

$$p^{>\lambda} = p \restriction \{(\kappa,\alpha,\beta) \mid \kappa > \lambda\}.$$

Now $p = p^{\leqslant\lambda} \cup p^{>\lambda}$. Let $\mathbb{P}^{\leqslant\lambda} = \{p^{\leqslant\lambda} \mid p \in \mathbb{P}\}$ and $\mathbb{P}^{>\lambda} = \{p^{>\lambda} \mid p \in \mathbb{P}\}$. Now $\mathbb{P} = \mathbb{P}^{\leqslant\lambda} \times \mathbb{P}^{>\lambda}$.

2.3. **Note.** *$\mathbb{P}^{>\lambda}$ is λ-closed. (Exercise)*

2.4. **Note.** *$\mathbb{P}^{\leqslant\lambda}$ satisfies the λ^+-c.c. (Exercise)*

2.5. **Lemma** (Gap Lemma)**.** *Suppose that λ is a regular cardinal, Q' is a λ-closed forcing and Q is a λ-c.c. forcing. Then for every Q'-generic H' and Q-generic H, every function $f\colon \lambda \to V$ in $V[H' \times H]$ is already in $V[H]$.*

Proof. Exercise. □

3. Finishing the proof and the Singular Cardinal Hypothesis (Video lecture 3)

Continuation of the proof...

Suppose now that some cardinal κ is not preserved by $\mathbb{P}$. Then there is a regular cardinal λ in $V[G]$ and a cofinal function $f\colon \lambda \to \kappa$. Since λ is regular in $V[G]$, it is regular also in V. Now splitting $\mathbb{P}$ as $\mathbb{P} = \mathbb{P}^{\leqslant\lambda} \times \mathbb{P}^{>\lambda}$ we can also split the $\mathbb{P}$-generic G as $G = G^{\leqslant\lambda} \times G^{>\lambda}$. Now by the Gap Lemma f is already in $V[G^{\leqslant\lambda}]$. But $\mathbb{P}^{\leqslant\lambda}$ has λ^+-c.c. so it cannot add f, a contradiction.

3.1. **Exercise.** $\mathbb{P}^{>\lambda}$ is λ-closed.

3.2. **Exercise.** $\mathbb{P}^{\leqslant\lambda}$ satisfies the λ^+-c.c.

To finish the proof we need to show that $(2^\lambda)^{V[G]} = F(\lambda)^V$ for all $\lambda \in A$. Note that since $\mathbb{P}$ preserves cardinals, $F(\lambda)^V$ is a cardinal in $V[G]$. Again by the Gap Lemma, any subset of λ that is in $V[G]$ is already in $V[G^{\leqslant\lambda}]$. So

$$(2^\lambda)^{V[G]} = (2^\lambda)^{V[G^{\leqslant\lambda}]}$$

. We know that $|p^{<\lambda}| = F(\lambda)$ by the GCH in V, so $(2^\lambda)^{V[G^{\leqslant\lambda}]} \leqslant F(\lambda)^\lambda = F(\lambda)$ again by the GCH in V.

But $(2^\lambda)^{V[G^{\leqslant\lambda}]}$ is $\geqslant F(\lambda)$ because the λ-coordinate of $\mathbb{P}$ codes the Cohen forcing for adding $F(\lambda)$ many subsets of λ.

22min 00sec mark Lecture 3

3.1. **Why cannot we have singular cardinals in A?** Observe that the assumption that A is a subclass of regular cardinals is necessary. Otherwise suppose that $A = \{\aleph_n \mid n < \omega\} \cup \{\aleph_\omega\}$, $F(\aleph_n) = \aleph_{n+1}$ and $F(\aleph_\omega) = \aleph_{\omega_5}$. It is known, however, that if $\aleph_\omega$ is a strong limit (i.e. for all $\lambda < \aleph_\omega$, $2^\lambda < \aleph_\omega$), then $2^{\aleph_\omega} < \aleph_{\omega_4}$ (a PCF-theorem by Shelah). This of course makes forcing the continuum function to be F impossible. It is unnecessary, however, to appeal to a deep theorem as the PCF-theorem above. Let A be as above and $F(\aleph_n) = \aleph_{\omega+1}$ for all n, and $F(\aleph_\omega) = \aleph_{\omega+2}$. These A and F satisfy the assumptions of the Easton's theorem except for one singular cardinal being in the set A. It is impossible, however, to force F to be the continuum function, because by Corollary 1.9 if $F(\aleph_n) = \aleph_{\omega+1}$ for all n, then $F(\aleph_\omega) = \aleph_{\omega+1}$.

3.2. **Singular Cardinal Hypothesis.** The Singular Cardinal Hypothesis (SCH) states that for every singular cardinal κ, if $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa^{\mathrm{cf}(\kappa)} = \kappa^+$. It obviously holds when GCH holds.

3.3. **Note.** *SCH holds in Easton's model.*

Proof. Suppose that κ is singular in $V[G]$. By the Gap Lemma, every function $f\colon \mathrm{cf}(\kappa) \to \kappa$ which is in $V[G]$ is contained in $V[G^{\leqslant \mathrm{cf}(\kappa)}]$. In $V[G]$, $2^{\mathrm{cf}(\kappa)} < \kappa$, so $F(\kappa)^{\mathrm{cf}(\kappa)} < \kappa$. So $|p^{\leqslant \mathrm{cf}(\kappa)}|^\kappa = F(\mathrm{cf}(\kappa))^\kappa = \kappa^+$ □

4. How to violate SCH? (Video lecture 4)

Is it possible to violate SCH? Historically, a weaker question was asked earlier: Is it possible that $V \neq L$? In 1961 Dana Scott used the method of elementary embeddings to prove the following.

4.1. **Theorem** (Scott). *If there is a measurable cardinal, then $V \neq L$.*

Proof. Recall κ is measurable if and only if there exists an elementary embedding $j\colon V \to M$ where $M \cong \mathrm{Ult}(V, U)$ is an utraproduct over an ultrafilter U such that the criticial point of j is κ. This is equivalent to having a κ-complete and normal ultrafilter on κ. Then the following properties hold:

(1) $j(\kappa) > \kappa$,
(2) ${}^{<\kappa}M \subset M$,

(3) $V_\kappa = (V_\kappa)^M$, where $(V_\kappa)^M$ is V_κ as interpreted in M.
(4) $2^\kappa \leqslant (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$.
(5) $(\kappa^+)^M = \kappa^+$,
(6) If θ is a strong limit cardinal, then $j(\theta) = \theta$ and $\mathrm{cf}(\theta) \neq \kappa$, then $j(\theta) = \theta$,
(7) $U \notin M$, ${}^\kappa M \not\subset M$.

Suppose towards a contradiction that $V = L$ and κ is the first measurable cardinal in V and let $j\colon V \to M$ witness that, where $M \cong \mathrm{Ult}(V, U)$. Since $V = L$ is a first-order statement that holds in V, it also holds in M and since M is an inner model of V, we have $V = M = L$. The statement $\varphi(\kappa)$ saying that "κ is the first measurable cardinal" is a first-order property and since $V \models \varphi(\kappa) \iff M \models \varphi(j(\kappa))$ we have a contradiction, because $V = M$ and $j(\kappa) > \kappa$ by the above.

4.2. **Corollary.** *There are no measurable cardinals in L.*

This gives rise to the so-called "small" large cardinals, i.e. those cardinals that are so large that their consistency is stronger than the consistency of ZFC, but which can still exist in L.

4.3. **Theorem** (Scott)**.** *A measurable cardinal cannot be the first one where GCH fails.*

Proof. Suppose that κ is measurable and $2^\lambda = \lambda^+$ for all $\lambda < \kappa$. Let $j\colon V \to M$ be an elementary embedding such that the critical point of j is κ. Then by elementarity M satisfies GCH below $j(\kappa)$ and since $\kappa < j(\kappa)$, we have $M \models 2^\kappa = \kappa^+$. But $2^\kappa \leqslant (2^\kappa)^M = (\kappa^+)^M = \kappa^+$. □

4.4. **Theorem** (Silver 1971/1974)**.** *A singular cardinal of uncountable cofinality cannot be the first at which GCH fails.*

How does this relate to Easton's model?

4.5. **Example.** (1) If $A = \{\alpha \leqslant \kappa \mid \alpha \text{ is regular}\}$ for some measurable κ, $F(\lambda) = \lambda^+$ for all $\lambda \in A, \lambda < \kappa$ and $F(\kappa) = \kappa^{++}$. Then the Easton's theorem applies to A and F, so κ is the first cardinal where GCH fails. **Note:** **[I think there is a problem with this example. The set A includes all singular cardinals below κ which is against the assumption of the Easton theorem. On the other hand, if A consists only of regulars below κ, then one should separately show that in the Easton extension $2^\lambda = \lambda^+$ for all singular $\lambda < \kappa$. This follows from the Silver's theorem for singulars with uncountable cardinality, but there are**

still a lot of ω-cofinal singulars below κ. This seems to me a good place to make a series of exercises!]

□

5. Proof of Silver's theorem (Video lecture 5)

Recall the Theorem:

5.1. **Theorem** (Silver 1971/1974). *A singular cardinal of uncountable cofinality cannot be the first at which GCH fails.*

Proof. Let κ be a singular cardinal and $\lambda = \mathrm{cf}(\kappa) > \aleph_0$. An increasing function between ordinals $f\colon \alpha \to \beta$ is *continuous* if it is continuous with respect to the order topology, i.e. if $A \subset \alpha$ and $\sup A = \gamma < \alpha$, then $\sup f[A] = f(\gamma)$. Let $h\colon \lambda \to \kappa$ be continuous increasing and cofinal. (A function like this can be found as follos. Let $g\colon \lambda \to \kappa$ be any cofinal function which exists by the assumption $\lambda = \mathrm{cf}(\kappa)$. By enumerating $g[\lambda]$ we can assume that g is increasing. Now by induction on $\alpha < \lambda$ define $h(\alpha)$ be the smallest element of $g[\lambda] \setminus \sup\{h(\beta) \mid \beta < \alpha\}$ if α is successor and $h(\alpha) = \sup_{\beta<\alpha} h(\beta)$ otherwise.) Assume that GCH holds below κ, so $2^\lambda = \lambda^+$. Denote $\mu = \lambda^+$.

The main trick is to force μ to be countable. This can be done for example by using Levy's collapse. Let G be generic. In $V[G]$ consider

$$U = \{f\colon \lambda \to \lambda \mid f \in V, f \text{ is regressive}\}$$

The cardinality of U, as computed in $V[G]$, is $\aleph_0$. Our task now is to find an object D in $V[G]$, a filter on λ but an ultrafilter for subsets of λ in V. Call this a V-ultrafilter. Thus for every $A \subset \lambda$, if $A \in V$, then either $A \in D$ or $\lambda \setminus A \in D$. Further, we require D to have the property that every regressive $f \in V$is constant on a D-positive set. Note that D cannot be in V, but it can be in $V[G]$, because there are only countable many requirements as $(\lambda^+)^{V[G]} = \aleph_0$.

Enumerate $U = \{f_i\colon i < \omega\}$. By induction on i, we build a decreasing sequence $\langle X_i \mid i < \omega\rangle$ of V-stationary subsets of λ such that $f_i \restriction X_i$ is constant. (Use Fodor's lemma at every step.) We are using the fact that $\lambda > \aleph_0$ in V. Define D in $V[G]$ to be

$$D = \{X \subset \lambda \mid X \in V \text{ and } X \supset X_i \text{ for some } i\}.$$

Since (X_i) was chosen to be a decreasing sequence, D is a filter. In fact, it is a V-ultrafilter which contains all stationary sets in V. Form in $V[G]$ the ultrapower $\mathrm{Ult}(V, D)$. Let $j\colon V \to \mathrm{Ult}(V, D)$ be an embedding. We can easily see that the critical point of j is λ. Now note that $j(h)$ is a continuous function from $j(\lambda)$ to $j(\kappa)$ by elementarity. So we should obtain the same contradiction as in the proof of the Scott's theorem (Theorem 4.3). $\square$

The above proof works for κ of uncountable cofinality. However, is the assumption $|\lambda| > \aleph_0$ necessary? It turns out it is. Silver himself found a model of set theory in which GCH fails for the first time at a singular cardinal of countable cofinality.

5.2. **Theorem** (Silver). *Modulo large cardinals, it is consistent that the first cardinal where GCH fails is a singular cardinal κ with $\mathrm{cf}(\kappa) = \aleph_0$.*

This is proved by forcing starting with a model which contains large cardinals. There is also an improvement to this theorem which does not tell us anything about the particular cardinal of countable cardinality.

5.3. **Theorem** (Magidor 1971). *Modulo large cardinals, it is consistent that $\aleph_\omega$is the first cardinal which fails GCH.*

The exact large cardinal strength of this theorem has been later established by Gitik.

5.4. **Theorem** (Gitik 1988). *$\aleph_\omega$ is the first cardinal failing GCH is equiconsistent with the existence of κ such that $o(\kappa) = \kappa^{++}$.*

The notation $o(\kappa) = \kappa^{++}$ means that κ is a measurable cardinal with a certain property. Mitchell **REFS** introduced an order that applies to measurable cardinals. $o(\kappa) = \kappa^{++}$ means that κ is a measurable cardinal of Mitchell-order κ^{++}. We will not define the Mitchell-order, but we will note that $o(\kappa) < o(\kappa')$ implies that κ' is of higher consistency strength than κ.

6. Prikry Forcing (Video lecture 6)

Prikry forcing preserves all cardinals, but does not preserve all cofinalities. It requires a measurable cardinal, whose cofinality will be changed to ω.

Let κ be a measurable and let $\mathcal{D}$ be a $(< \kappa)$-complete normal ultrafilter on κ. Let

$$\mathrm{Pr}(\mathcal{D}) = \{(s, A) \mid s \in [\kappa]^{<\omega}, A \in \mathcal{D}, A \subset \kappa \setminus (\max(s) + 1)\}.$$

These conditions are ordered as follows. $(s, A) \leqslant (t, B)$ if and only if s is an initial segment of t, $A \supset B$ and $t \setminus s \subset A$.

Observations.

(1) $\mathrm{Pr}(\mathcal{D})$ adds generically an ω-sequence cofinal in κ. (Why? Let $G = \{(s_i, A_i) \mid i < \kappa\}$ be $\mathrm{Pr}(\mathcal{D})$-generic and let $f = \bigcup\{s \colon \exists A(s, A) \in G\}$. Then f is cofinal, because $\mathcal{D}_\alpha = \{(s, A) \mid \exists \beta \in \sigma \setminus \alpha\}$ is dense for all $\alpha > \kappa$. Similarly order-type of f is ω).

(2) If $A, B \in \mathcal{D}$, $s \in [\kappa]^{<\omega}$, $\min(A), \min(B) > \max(s)$, then (s, A) and (s, B) are compatible: the common extension is $(s, A \cap B)$.

(3) By the above, any antichain will consist of conditions with different stems and since $|[\kappa]^{<\omega}| = \kappa$, $\mathrm{Pr}(\mathcal{D})$ satisfies κ^+-c.c.

(4) $\mathrm{Pr}(\mathcal{D})$ does not add bounded subsets to κ. This implies that cardinals below κ are preserved. In particular κ does not collapse to $\aleph_0$.

How to prove (iv)? $\mathrm{Pr}(\mathcal{D})$ is not $\aleph_0$-closed: it is easy to construct a sequence $(s_i, A_i)_{i<\omega}$ such that $|s_i|$ increases in i. So we need something else.

6.1. **Prikry Lemma.** Let us start by stating the Rowbottom theorem;

6.1. **Theorem** (Rowbottom theorem)**.** *If λ is a measurable cardinal, U a measure on λ and $g \colon [\lambda]^{<\omega} \to 2$, then there is $H \in U$ such that $g \restriction [H]^{<\omega}$ is constant.* □

If p is a forcing condition and φ is a sentence in the forcing language, we say that p *decides* φ, if p either forces that φ is true or that φ is false. This is denoted by $p \| \varphi$. Note that for any sentence of the forcing language and any p we have that

$$p \Vdash \text{“}\varphi \text{ is either true or false”}.$$

6.2. **Lemma** (Prikry Lemma)**.** *Suppose that $(s, A) \in \mathrm{Pr}(\mathcal{D})$ and φ is a sentence in the forcing language. Then we can extend (s, A) to decide φ without extending the stem: there exists $(t, B) \geqslant (s, A)$ such that $(t, B) \| \varphi$ and $s = t$.*

Note: It seems to me that there is a problem in the proof you gave in the lecture (Lecture 6, 30min→). Namely if you define f in this way:

$$f(t) = \begin{cases} 1, & \textbf{if } \exists X \in \mathcal{D}((s \cup t, X) \Vdash \varphi) \\ 0, & \textbf{otherwise.} \end{cases}$$

then f can be constant 0 (if φ is false) in which case the Claim cannot be true, because the Claim says that f is constant 1 on a large set. I modify the proof so that it works (but tell me if I am wrong!)

Proof. First note that the Rowbottom theorem applies even if 2 is replaced by 3 and λ is replaced by any $\mathcal{D}$-large subset of λ (Exercise).

Now let $f\colon [A \setminus (\max(s)+1)]^{<\omega} \to 3$ be defined as follows: for each $t \in [A \setminus (\max(s)+1)]^{<\omega}$ let

$$f(t) = \begin{cases} 2, & \text{if } \exists X \in \mathcal{D}((s \cup t, X) \Vdash \varphi) \\ 1, & \text{if } \exists X \in \mathcal{D}((s \cup t, X) \Vdash \neg\varphi) \\ 0, & \text{otherwise.} \end{cases}$$

By the Rowbottom theorem there is now $B \subset A$ such that $B \in \mathcal{D}$ and f is constant on $[B]^{<\omega}$. If we show that the value of f on B is either 1 or 2, then we are done. So let us show that it cannot be 0. If it is 0, then in particular $f(\varnothing) = 0$, so (s, B) does not decide φ. Now there exist $X_1, X_2 \in \mathcal{D}$ and $s_1, s_2 \in [B]^{<\omega}$ such that (s_1, X_1) and (s_2, X_2) are extensions of (s, B), $(s_2, X_2) \Vdash \varphi$ and $(s_1, X_1) \Vdash \neg\varphi$. Let $t_2 = s_2 \setminus s$ and $t_1 = s_1 \setminus s$. Then this means that $f(t_2) = 2$ and $f(t_1) = 1$ which is a contradiction, because by the definition of extension $t_2, t_1 \subset B$. □

7. Applications of Prikry forcing and further research (Video lecture 7)

We want now to prove that Prikry forcing does not add bounded subsests of κ.

7.1. **Definition.** An extension $(s, A) \leqslant (t, B)$ is *pure*, denoted by $(s, A) \leqslant^* (t, B)$, if $s = t$.

Observation. $\leqslant^*$ is $< \kappa$-closed, because $\mathcal{D}$ is $< \kappa$-closed: If (s, A_i) is a chain $i < \alpha < \kappa$, then $(s, \bigcap_{i<\alpha} A_i)$ is a common extension.

7.2. **Theorem.** *Prikry forcing does not add bounded subsets to κ.*

Proof. Suppose $\lambda < \kappa$. We need to show that any name τ for a subset of λ is decided on a dense set of conditions. That is, every (s, A) has an extension (t, B) such that $(t, B) \Vdash (\tau = X)$ for some $X \in V$.

So let τ and (s, A) are such that $(s, A) \Vdash \tau \subset \lambda$. By induction on $\alpha < \lambda$ choose A_α such that (s, A_α) is an increasing sequence and $(s, A_{\alpha+1}) \| (\alpha \in \tau)$:

- $A_0 = A$,
- for limit α, $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$,
- for $\alpha = \beta + 1$, apply Prikry Lemma to A_β and "$\alpha \in \tau$".

Let $B = \bigcap_{\alpha<\lambda} A_\alpha$, so $(s, B) \geqslant (s, A_\alpha)$ for all α. Therefore

$$(s, B) \Vdash \tau = \{\alpha \mid (s, A_\alpha) \Vdash (\alpha \in \tau)\}$$

. □

Conclusion: In the extension $\kappa > \mathrm{cf}(\kappa) = \aleph_\omega$, but all cardinals are preserved and κ is a strong limit.

SO, how to fail SCH? Start with a model where κ is measurable and GCH holds below κ (possible by a theorem of Solovay). Then force $2^\kappa > \kappa^+$ while keeping κ measurable. This is not trivial, because we know from a theorem of Scott (Theore 4.3) that a measurable cardinal cannot be the first on where GCH fails. So we have to make $2^\alpha > \alpha^+$ for unboundedly many $\alpha < \kappa$. Such forcing will preserve the measurability of κ while making $2^\kappa > \kappa^+$. Then we apply Prikry forcing to make κ singular. Since Prikry forcing does not add bounded subsets of κ, we still have $2^\alpha < \kappa$ for all $\alpha < \kappa$ in the extension, so in particular $2^{\mathrm{cf}(\kappa)} < \kappa$ and $2^{<\kappa} = \kappa$. On the other hand Prikry forcing preserves cardinals, so

$$\kappa^{\mathrm{cf}(\kappa)} = (2^{<\kappa})^{\mathrm{cf}(\kappa)} = 2^\kappa > \kappa^+$$

the last equality holds by Lemma 1.8.

So the only remaining question is how to preserve measurability. More generally, how to preserve combinatorial properties of κ?

17min 00sec mark Lecture 7

"Further research"

"Research plan..."

Motivating question...

A universal graph is a graph into which all graphs of the same size embed. Let $P(\kappa)$ denote the property that there exists a universal graph on κ. How to preserve $P(\kappa^+)$ while forcing failure of SCH at κ?

More generally: Get SCH to fail at κ and simultaneously have some non-trivial combinatorics on κ^+.

Motivation: Explore how much a singular cardinal κ with $\mathrm{cf}(\kappa) = \aleph_0$ is similar to ω.

For example it has been shown (Dzamonja-Väänänen **REFS**) that the logic $L_{\kappa\kappa}$ is very similar to the first-order logic $L_{\omega\omega}$ for such cardinal κ.

Test problem: It is known (Shelah-Mekler **REFS**) that it is consistent that $2^\omega = \omega_2$ and that there is a universal graph on ω_1. Can we get the same result on a cardinal κ with $\mathrm{cf}(\kappa) = \omega$, i.e. is it possible that there is a universal graph on κ^+ while $2^\kappa > \kappa^+$ and $2^{<\kappa} = \kappa$?

In the case of ω this is done using an ω_2-long iteration. In the case of κ singular this is impossible, because in order to fail SCH we must start with a large cardinal. See videolecture for an explanation with pictures! :)

Homework: Suppose that $V \subset V'$, $\mathcal{D}$ is an ultrafilter on κ in V, $\mathcal{D}'$ is an ultrafilter on κ in V' and $\mathcal{D} \subset \mathcal{D}'$. Then (1) every $\mathrm{Pr}(\mathcal{D})$-condition is also a $\mathrm{Pr}(\mathcal{D}')$-condition and (2) $\perp$ in $\mathrm{Pr}(\mathcal{D}')$ is preserved in $\mathrm{Pr}(\mathcal{D})$ for elements of $\mathrm{Pr}(\mathcal{D})$. (3) Antichains in $\mathrm{Pr}(\mathcal{D})$ have a "natural interpretation in $\mathrm{Pr}(\mathcal{D}')$. (4) Nice names in $\mathrm{Pr}(\mathcal{D})$ give rise to nice names in $\mathrm{Pr}(\mathcal{D}')$.

What about other combinatorics than the universal graph?

References

[1] András Hajnal and Peter Hamburger. *Set theory*, volume 48. Cambridge University Press, 1999.
[2] T. Jech. *Set Theory*. Springer-Verlag Berlin Heidelberg New York, 2003.